Summer 2016 Written Comprehensive Exam
Opti 501

System of units: MKSA

A monochromatic plane electromagnetic wave propagates in free space along the z-axis. The beam is linearly polarized along the x-axis, having E-field amplitude $E_0\hat{x}$ and H-field amplitude $H_0\hat{y}$. In general, the field amplitudes are complex-valued; for instance, $E_0 = |E_0|\exp(i\varphi_0)$.

3 Pts a) In terms of the frequency ω_0 of the oscillations, the speed c of light in vacuum, the impedance Z_0 of free space, and the E-field amplitude and phase, $|E_0|$ and φ_0, write expressions for the *real-valued* E and H fields as functions of the space-time coordinates (x, y, z, t).

3 Pts b) Determine the rate of flow of electromagnetic energy (per unit cross-sectional area per unit time) at a given instant of time, say, $t = t_0$, at two points P_1 and P_2 located at $(x, y, z) = (0, 0, z_1)$ and $(x, y, z) = (0, 0, z_2)$, where z_1 and z_2 are two arbitrary points along the z-axis.

4 Pts c) According to your answer to part (b), the rate of flow of energy at (P_1, t_0) differs from that at (P_2, t_0) — unless the distance $z_2 - z_1$ happens to be a half-integer-multiple of the wavelength $\lambda_0 = 2\pi c/\omega$. Identify where the missing energy has gone. (You must provide a detailed answer that accounts for the missing energy with perfect accuracy.)

Hint: The following identities will be helpful:

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin x + \sin y = 2\sin\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right); \qquad \sin x - \sin y = 2\sin\left(\tfrac{x-y}{2}\right)\cos\left(\tfrac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\tfrac{x+y}{2}\right)\cos\left(\tfrac{x-y}{2}\right); \qquad \cos x - \cos y = -2\sin\left(\tfrac{x+y}{2}\right)\sin\left(\tfrac{x-y}{2}\right)$$

System of units: MKSA

A plane electromagnetic wave of frequency ω arrives at normal incidence at the interface between a transparent dielectric medium of refractive index n_0 and an absorptive medium specified by its complex refractive index $n + \mathrm{i}\kappa$. The incident beam is linearly polarized, having E-field amplitude $E_0\hat{x}$ and H-field amplitude $H_0\hat{y}$. Assume $\mu(\omega) = 1$ for both media, the speed of light in vacuum is $c = 1/\sqrt{\mu_0\varepsilon_0}$, and the impedance of free space is $Z_0 = \sqrt{\mu_0/\varepsilon_0}$.
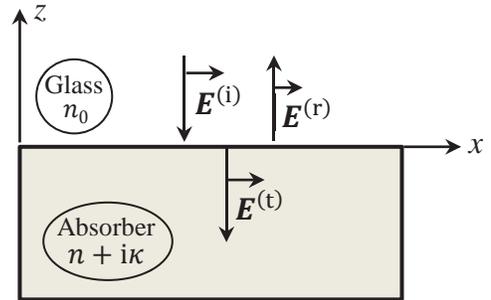
2 Pts a) Write expressions for the E and H fields in the semi-infinite medium of incidence (transparent dielectric), and in the semi-infinite absorptive substrate.

2 Pts b) Match the boundary conditions at the interface between the two media (located at $z = 0$), then proceed to determine the Fresnel reflection and transmission coefficients at the boundary.

3 Pts c) Use the *time-averaged* Poynting vector $\langle S(r,t) \rangle$ to determine the rate of flow of electromagnetic energy (per unit cross-sectional area, per unit time) in the incident, reflected, and transmitted beams at the interface between the two media (i.e., at $z = 0^+$ and $z = 0^-$).

3 Pts d) Confirm that the difference between the incident and reflected optical energies is in fact equal to the energy captured within the absorptive substrate.

Fall 2016 Written Comprehensive Exam and Qualify Exam
OPTI 502

A 12X Keplerian telescope with an Angular Field of View of +/- 0.1 degree in object space is constructed out of two thin lenses.

Notes: 1) Assume thin lens in air.
      2) Only the magnitude of a magnification or magnifying power may be given.
      3) On some quantities, only the magnitude of the quantity is provided. The proper sign convention must be applied.
      4) Light travels from left to right.
      5) Provide your answers in a neat and orderly fashion.

Part 1) The focal length of the eye lens is 30 mm. The telescope has 24 mm entrance pupil diameter. Determine the f-number of the objective lens, which serves as the system stop, and the telescope length.

The f-number of the objective lens is _____.
The telescope length is _____.

Part 2) Determine the eye relief of the telescope.

The eye relief is _____.

Part 3) In order for the system is to be unvignetted over the entire field of view, a field lens is inserted at the intermediate image plane. Determine the diameter of the field lens, the resulting

magnifying power of the telescope (with the proper sign), and the length of the telescope including the field lens.

The diameter of the field lens is _____.
The magnifying power of the unvignetted telescope is _____.
The unvignetted telescope length is _____.

Part 4) A thin lens is added at the exit pupil of the telescope in order to convert the telescope into a camera system. A detector is located at +50 mm to the right of this imaging lens. Determine the focal length of the thin lens. Also determine the focal length of the telescope combined with the imaging lens.

The focal length of the thin imaging lens is _____.
The focal length of the combined system is _____.

Part 5) For the system of Part 4, determine the required dimensions of the detector needed to cover the entire field of view, which is +/- 0.1 degree (in object space). The detector has a fixed 4 by 3 aspect ratio rectangular format.

The size of the detector area is _____by_____.

Part 6) The telescope objective lens is made out of N-BK7 (glass code: 517642). What is the refractive index (n_d) of N-BK7? What is the longitudinal chromatic aberration induced by the objective lens?

The refractive index of N-BK7 is _____.
Longitudinal chromatic aberration of the objective lens is _____.

Fall 2016 Written Comprehensive Exam and Qualify Exam
OPTI 502

A telephoto lens (shown below) consists of a positive lens with a focal length f_1 followed by a negative lens with a focal length f_2. The distance between two lenses is d. The key advantage of this configuration is that the focal length f of the system is larger than the overall system length L. The reduction factor k of the length with respect to the focal length is

$$k = \frac{L}{f}$$
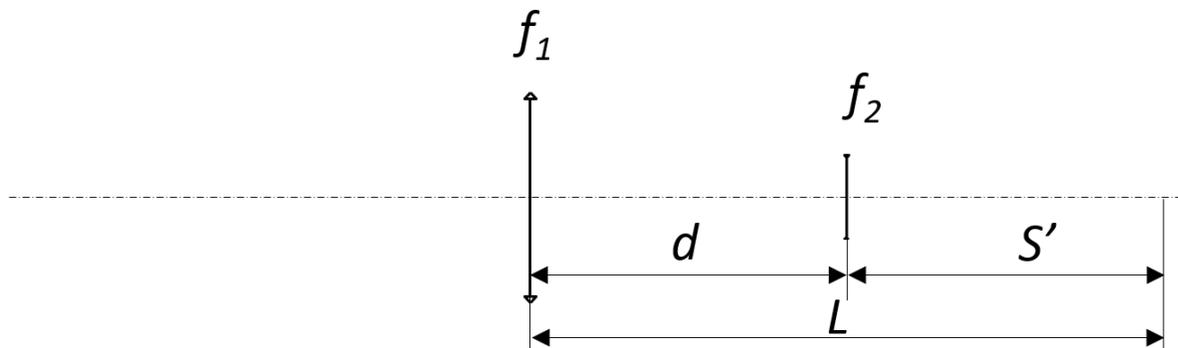
1. Given f, d, and k, derive f_1 and f_2. (**4 pts**)
2. Derive the back focal distance S'. (**1 pt**)
3. To achieve the telecentric condition in image space, drive the location of aperture stop. (**3 pts**)
4. For this telecentric configuration, sketch the system showing the marginal and chief rays and identifying the principal planes, and the aperture stop. Assume an object at infinity and an image height h'. (**2 pts**)

Fall 2016 Written Comprehensive Exam and Qualify Exam
OPTI 505R

Consider a Twyman-Green interferometer arranged to measure the power spectrum of a collimated laser beam. Both interfering beams in the observation path have the same irradiance. A small portion of a single fringe is measured as a function of OPD on a silicon detector, whose output is electrical current i with $i \propto I_{tot}$, and I_{tot} is the irradiance at the measurement point on the fringe. OPD is changed by moving one of the interferometer mirrors.

    a.) (2pt) Make a drawing of the instrument, labeling significant components.

    b.) Consider the power spectrum given by:

$$g(v) = C \left[\text{gaus}\left(\frac{v}{\Delta v_{mode}} \right) * \text{comb}\left(\frac{v}{c/2L} \right) \right] \text{gaus}\left(\frac{v - v_0}{\Delta v} \right),$$

where (*) denotes one-dimensional convolution, $L = 1$m, $v_0 = 6.147 \times 10^{14}$ Hz (corresponds to $\lambda_0 = 488$ nm average wavelength), $\Delta v = 5$ GHz , $\Delta v_{mode} = 3$ MHz (individual longitudinal mode linewidth), c is speed of light in a vacuum (3×10^8 m/sec), and C is a normalization constant.

i) (2pts) Draw a graph showing the current signal i as a function of OPD between the two arms of the interferometer with OPD between 0 and 100mm. Due to rapid oscillations, you may not be able to show all detail in your graph. In that case, the envelope would be sufficient.

ii) (1pt) What is the period of the rapid oscillations in (i), in terms of the mirror position used to change OPD?

iii) (1pt) How many individual modes oscillate within the envelope of Δv ? Explain your answer.

iv) (1pt) What is the coherence length of this laser?

    c.) It is desired to operate the laser as a single-frequency source, and a Fabry Perot etalon is designed for that purpose.

i) (1pt) What is the required minimum mirror reflectivity if both mirrors have the same value of reflectivity and only one individual mode is transmitted through the cavity? Explain your answer.

ii) (1pt) What is the maximum separation of the cavity mirrors, so that modes are uniquely identified?

iii) (1pt) What is the coherence length of the single-frequency laser?

You may find the following information useful:

$$\text{gaus}\left(\frac{x}{b}\right) = e^{-\pi(x/b)^2}$$

$$\mathbf{F}_\xi\left[\text{gaus}(x)\right] = \text{gaus}(\xi)$$

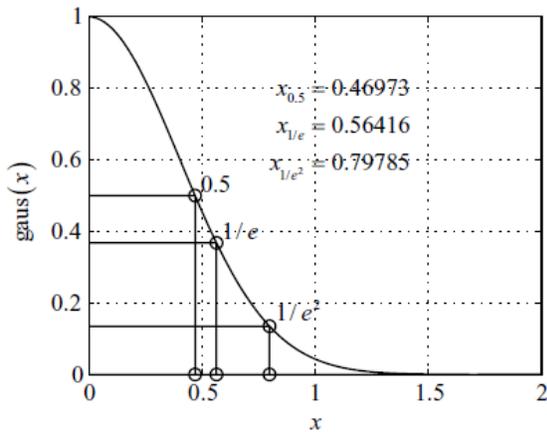$$\int_\infty \text{gaus}\left(\frac{x}{b}\right) dx = |b|$$

$$\mathbf{F}_\xi\left[\text{comb}(x/b)\right] = b\,\text{comb}(b\xi) = \sum_m \delta(\xi - m/b)$$

$$\mathcal{F} = \frac{\pi\sqrt{R}}{1-R}$$

$$\left|\frac{\lambda}{\Delta\lambda_{RES}}\right| = m\mathcal{F} = \frac{m\pi\sqrt{R}}{1-R}.$$

$$\Delta\lambda_{FSR} = \frac{\lambda_1}{m_2} \approx \frac{\overline{\lambda}^2}{2n_2 d \cos\theta_2},$$

For $ax^2 + bx\_ + c = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Fall 2016 Qualifying Exam Question

505R

Consider a square wire mesh placed across a square
aperture, as shown at the right. The aperture is
opaque outside of the square with dimension D. A
cross section of the amplitude transmittance t (same
along x-axis and y-axis) is also plotted, along with the
critical dimensions. Assume this aperture is
illuminated in air with uniform, collimated, coherent
light of wavelength λ and irradiance I_s. Also assume
$\lambda \ll d < L \ll D$.
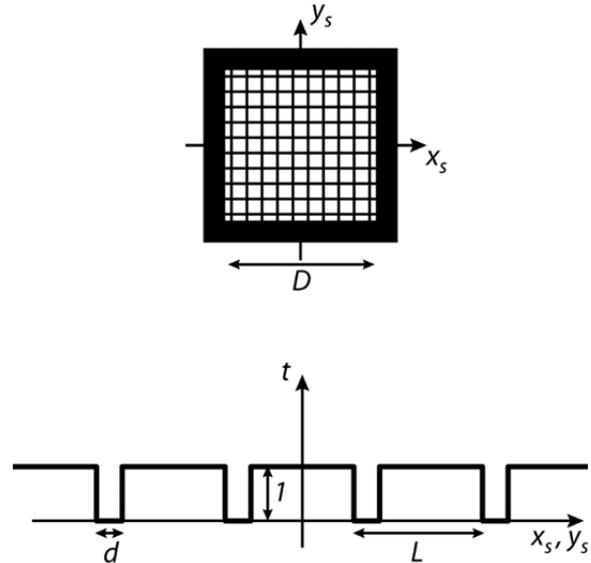
a) Write the mathematical expression for the
   irradiance of the pattern in the far-field.

b) Sketch the irradiance of the pattern along the x-axis in the far-field and annotate it with the
   approximate length scales of the various features in the pattern.

c) Approximately, at what distance z_0 from the screen will the Fraunhofer approximation be valid?
   Will different features of the aperture produce Fraunhofer patterns at different distances? If so,
   what are those distances?

d) A thin lens of focal length f is placed against the screen. Describe in words how the irradiance
   pattern observed a distance f after the lens compares to the pattern in part (b), with regard to
   the overall shape of the pattern and length scales involved.

Reference information:

Fraunhofer integral:

$$-j\frac{e^{jkz_0} \, e^{\frac{jk}{2z_0}(x_0^2+y_0^2)}}{\lambda z_0} \iint U_S^+(x_s, y_s) \, \exp\left[-j\frac{2\pi}{\lambda z_0}(x_s x_0 + y_s y_0)\right] dx_s \, dy_s$$

Fourier pairs:

$f(x)$	$g(\xi) \equiv \int_{-\infty}^{\infty} f(x)e^{-j2\pi\xi x} \, dx$
$\text{comb}_T(x) \equiv \sum_{m=-\infty}^{\infty} \delta(x - mT)$	$\frac{1}{T}\text{comb}_{\frac{1}{T}}(\xi)$

$\text{rect}_A(x) \equiv \begin{cases} 0 \text{ for } |x| > A/2 \\ 1/2 \text{ for } |x| = A/2 \\ 1 \text{ for } |x| < A/2 \end{cases}$	$A \, \text{sinc}(A\xi) \equiv A \dfrac{\sin(\pi A\xi)}{\pi A\xi}$

Fall 2016
OPTI 511 and OPTI 544

Here you are asked to provide energy eigenfunction and/or energy eigenvalue solutions to the time-independent Schrödinger equation (TISE) for a particle of mass m subject to various potential energy functions V. For the different cases, define any parameters or quantum numbers that you use if they are not otherwise defined or used in the question, and specify their values. **You may neglect normalization coefficients for all energy eigenfunctions.**

(a) For a free particle in the one-dimensional coordinate x, $V = 0$ everywhere. The TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_E(x) = E\psi_E(x).$$

Specify the values of E and the corresponding functions $\psi_E(x)$ that solve this equation.

(b) Consider a one-dimensional TISE where the potential energy $V(x)$ is defined by

$$V(x) = \begin{cases} 0 & \text{for } 0 \geq x \geq L \\ \infty & \text{otherwise.} \end{cases}$$

Specify the energy eigenvalues E_n and the energy eigenfunctions $\psi_n(x)$ associated with this potential, for any integer n greater than or equal to 1.

(c) For a one-dimensional problem with potential energy $V(x) = \frac{1}{2}m\omega^2 x^2$, where ω is an angular frequency and x is the position coordinate, the TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n(x) + \frac{1}{2}m\omega^2 x^2\psi_n(x) = E_n\psi_n(x).$$

Specify the energy eigenvalues E_n, giving an expression that is valid for all integers n greater than or equal to zero. You should not attempt to solve this TISE from scratch.

(d) For the case defined in **(c)**, sketch the **probability density** distributions vs x for the ground state and first two excited states. What name is given to the full set of energy eigenfunctions?

(e) Consider an electron bound within the three-dimensional central potential $V(r) = \frac{-e^2}{4\pi\epsilon_0 r}$, where r is the distance away from the origin of the coordinate system, e is the quantum of electric charge, ϵ_0 is the permittivity of free space (also called the vacuum permittivity), and m in the TISE is approximately equal to the electron's mass. Give an expression for the energy eigenvalues (and the range of quantum numbers) of the TISE. Give numerical values for any constants used in your answer.

Fall 2016 Written Comprehensive Exam and Qualify Exam
OPTI 511R and OPTI 544

Consider a large ensemble of hydrogen atoms all initially in the ground state ψ_{100}. At time $t = 0$ a monochromatic and linearly polarized laser field of magnitude $E = \frac{1}{2}E_o e^{-i\omega t} + c.c.$ is turned on and interacts with all the atoms. The laser frequency ω_o is slightly detuned from the $n = 1$ to $n = 3$ transition, with detuning $\Delta = \omega - \omega_o$, where $|\Delta| \ll \omega_o$. In this problem, assume the electric dipole and rotating wave approximations are valid, and that the atoms are motionless (ie no Doppler broadening). See the bottom of this page for information you may find useful.

(a) Write down the general expression that gives the probability for finding the atom in the excited state as a function of time (in the absence of spontaneous emission). Be sure to clearly define any symbols used in your expression.

(b) Given a laser detuning of $\Delta = 2\pi \times 100$ MHz and an electric field strength of $E_o = (\frac{3\pi\hbar}{ea_o}) \times 10^8 \approx 11.6 \times 10^3 \left[\frac{J}{C \cdot m}\right]$, calculate the maximum possible fraction of atoms that can be excited to level $n = 3$ (ignoring spontaneous emission). Provide a numeric answer.

(c) Now assume the entire ensemble of atoms is in the state $\psi_{3,1,0}$. What wavelength(s) would be observed as the atom spontaneously decays down to its lowest energy state?

(d) Using the following expression for the Einstein A coefficient and the appropriate matrix elements of hydrogen, determine the *total* lifetime of this state (provide a numeric answer).

$A = \frac{\omega_0^3 |\vec{\wp}|^2}{3\pi\epsilon_0 \hbar c^3} = \left|\frac{\delta E}{E_1}\right|^3 \times \frac{|\vec{\wp}|^2}{e^2 a_0^2} \times 2.35 \times 10^9 \text{ s}^{-1}$, where δE is the energy difference between 2 states.

$\langle l = 0, m = 0 \,|\hat{\mathbf{r}}|\, l = 1, m = 0 \rangle = (0, 0, \sqrt{\frac{1}{3}})$

$\langle l = 0, m = 0 \,|\hat{\mathbf{r}}|\, l = 1, m = 1 \rangle = (-\sqrt{\frac{1}{6}}, -i\sqrt{\frac{1}{6}}, 0)$

$\langle l = 0, m = 0 \,|\hat{\mathbf{r}}|\, l = 1, m = -1 \rangle = (\sqrt{\frac{1}{6}}, -i\sqrt{\frac{1}{6}}, 0)$

$\langle n = 1, l = 0 \,|r|\, n = 2, l = 1 \rangle = 1.29 a_0$
$\langle n = 1, l = 0 \,|r|\, n = 3, l = 1 \rangle = 0.517 a_0$
$\langle n = 2, l = 0 \,|r|\, n = 3, l = 1 \rangle = 3.07 a_0$
$\langle n = 2, l = 1 \,|r|\, n = 3, l = 0 \rangle = 0.95 a_0$

(a)

Consider the density of states $g(\varepsilon)$ of the parabolic conduction band subbands in a GaAs quantum well. Without any derivation, plot $g(\varepsilon)$ including the first three subbands. If there are any elements in the plot that have the same length, clearly indicate these elements.

(5 points)

(b)

Consider again the conduction band density of states in a GaAs quantum well, but this time only the first subband (which is two-fold degenerate because of spin). Assume that the band energy is isotropic and, rather than parabolic, given by $\varepsilon_c(k) = \varepsilon_0 + c\,k^4$,

where c is a positive constant. Using the general formula $g(\varepsilon) = \dfrac{2}{A}\sum_{\vec{k}} \delta(\varepsilon - \varepsilon_c(k))$,

obtain an analytical expression of the density of states and plot your result, properly labeling the axes and all special points.

(5 points)

Consider a one-dimensional crystal with conduction band energy given by
$\varepsilon_c(k) = \varepsilon_0 + 2B\cos(ka)$, where $B = 1.5\,\text{eV}$ and the lattice constant of $a = 5\,\text{Å}$.
Determine the maximum electron speed (absolute value of velocity) and the corresponding wave vectors inside the first Brillouin zone. If you don't know the formula of the velocity, remember that it is analogous to the group velocity of light. Plot the velocity over the entire first Brillouin zone, properly labeling the axes and all special points. (You can use $\hbar = 0.658\,\text{meV ps}$.)

(10 points)

Fall 2016 Written Comprehensive Examination
OPTI 537

Consider imaging systems that are imaging coherent, monochromatic objects with wavelength λ. Fresnel diffraction for a system described by an ABCD matrix is given by:

$$u_{out}(\vec{r}) = \frac{-i}{B\lambda} \int_{-\infty}^{\infty} d^2 r_0 \, u_{in}(\vec{r_0}) \exp\left(\frac{i\pi}{\lambda B}[A|\vec{r_0}|^2 + D|\vec{r}|^2 - 2\,\vec{r}\cdot\vec{r_0}]\right) \qquad Eq\ (1)$$

where both $\vec{r}$ and $\vec{r_0}$ are 2 vectors in the x-y plane. (A selection of ABCD matrices is given in the table at the bottom of the page.)

   (a) Show that Equation 1 above simplifies to standard Fresnel transform when considering free-space propagation.
   (b) Now consider the simple imaging system below,
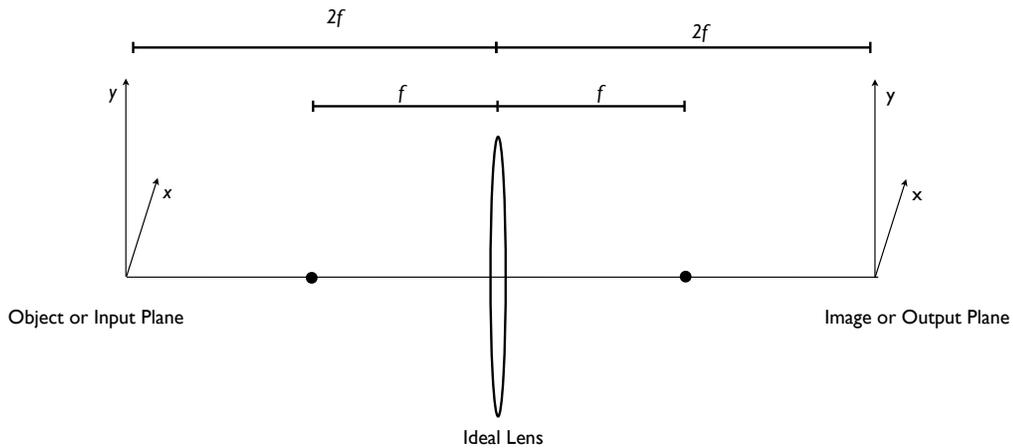
       Compute the ABCD matrix for this system and describe the difficulties of using this result with Equation 1.
   (c) Now, move the object plane to the left focal point and the image plane to the right focal point. Compute the ABCD matrix for this new system and derive an expression for the Fresnel diffraction for this imaging problem. Be sure to comment on the final form of your expression.
   (d) Use the ABCD matrices given in the table below to derive the ABCD matrix for a thick lens with radii of curvature R1 and R2, and center thickness t. Assume that the lens is in air. Describe the convention for the signs of R1 and R2.
   (e) For a convex lens, assume that |R1| = |R2|, and t=0 (to approximate a thin lens). Derive an expression for the focal length (or 1/f if you prefer) using the expression you determined above.

ABCD Matrices:

Propagation in a medium of constant refractive index	$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$	d = distance along optical axis
Refraction at a flat interface	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$	n1 = initial refractive index n2 = final refractive index
Refraction by a thin lens	$\begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$	F = focal length (f>0 means convex lens)
Refraction from a curved interface	$\begin{bmatrix} 1 & 0 \\ (n_1 - n_2)/Rn_2 & n_1/n_2 \end{bmatrix}$	R = radius of curvature n1 = initial refractive index n2 = final refractive index

Fall 2016 Written Comprehensive Examination
OPTI-537

Answer the following questions related to solid-state physics and semiconductor detectors. All parts weighted as indicated.

(a) (1 Point) Write the 3D time-independent Schrödinger equation for a single electron in a potential with periodicity defined by a Bravais lattice. Explain what assumptions are necessary to invoke the time-independent form of the Schrödinger equation to describe the electronic structure and related properties of crystals. Write a Fourier representation for the potential that expresses the periodicity, and show that it does so.

(b) (2 pts) Show how invoking Bloch's theorem and the Born-Von Karman boundary condition (1D is OK) leads to quantized momentum states. What is the separation between states for a 1D crystal of dimension A if the direct lattice constant has length a? How many electrons fit inside a single band in the 1st Brillouin zone of this 1D crystal?

(c) (2 pts) Then consider a 2D square lattice with primitive lattice constants a = b = 5 Å. Use the Ewald sphere construction to work out what the allowed elastic reflections (k_{out}) (expressing them in terms of $\hat{A}$ and $\hat{B}$) are if light has an incident k vector (in units of $Å^{-1}$)

$$k_{inc} = \frac{4\pi}{5}\hat{A} + \frac{\pi}{5}\hat{B}$$

where $\hat{A}$ and $\hat{B}$ are the unit vectors in reciprocal space corresponding to directions $\hat{a}$ and $\hat{b}$ in real space.

(d) (1pt) If a one-electron state of energy ε_1 has a probability of occupancy of .20 at some temperature T, what is the probability of occupancy (at the same temperature) of another one-electron state of energy $\varepsilon_2 = 2\varepsilon_1 - \varepsilon_F$ where ε_F is the Fermi energy?

(e) (2 pts) Make a sketch that shows the concentrations of majority and minority charge carriers versus temperature ranging from 0K to 500K for typical crystals of a P-type doped semiconductor. Annotate the plot with an explanation of what physical effects are being revealed.

(f) (2 pts) Make a sketch of the basic structure (cross-section) of a CMOS pixel array detector that explains how the device functions as a photodetector, where charge is stored, and how it is read out.

Fall 2016 Written Comprehensive Exam
Opti 546

This problem involves a plane-wave $E_0 e^{ikz}$ propagating along the z-axis that impinges upon a screen at $z = 0$ with field transmission $t(x')$, resulting in a field just after the screen $E(x', z = 0) = E_0 t(x')$. The screen and field are assumed homogeneous along the y-axis and we restrict the analysis to one transverse dimension. Then the field in the Fraunhofer region a distance L beyond the screen is given by

$$E(x, L) = \int_{-\infty}^{\infty} dx' E(x', 0) e^{-\frac{ikxx'}{L}},$$

where for simplicity a prefactor multiplying the integral has been omitted. The following tabulated integral may be of use

$$\int_{-\infty}^{\infty} ds\, e^{-isq - bs^2} = \sqrt{\frac{\pi}{b}} e^{-q^2/4b}.$$

(a - 2pts) First consider the case of a single Gaussian aperture with transmission $t(x') = e^{-(x'-x_0)^2/\sigma^2}$, x_0 being the position of the aperture center and σ is a measure of the aperture width. Using the information above derive an expression for the diffracted field $E(x, L)$ in the Fraunhofer region. As a check you should find that the diffracted field includes a Gaussian envelope factor e^{-x^2/w^2}, where $w = 2L/k\sigma$.

(b - 2pts) Next consider the case with two Gaussian apertures separated by a distance d with transmission $t(x') = [e^{-(x'-d/2)^2/\sigma^2} - e^{-(x'+d/2)^2/\sigma^2}]$. Use your result from part (a) to obtain an expression for the resulting transverse intensity profile $|E(x, L)|^2$.

(c - 1pt) Using your result from part (b) obtain an expression for the spacing Δ between adjacent zeros in the transverse intensity profile.

(d - 3pts) Assuming $d >> \sigma$ provide a sketch of $|E(x, L)|^2$ versus x indicating key features including the width of the transverse intensity profile and the positions of zeros in the intensity profile.

(e - 2pts) If the Gaussian apertures are replaced by narrow slits *explain* how the overall shape and structure of your plot from part (d) would change.

Fall 2016 Written Comprehensive Exam
Opti 546

This problem deals with the properties of the Laguerre-Gaussian (LG) modes of optical resonators. Recall that the mode frequencies of a two-mirror optical resonator are given by

$$\nu_{qp\ell} = \frac{c}{2L}[q + \frac{1}{\pi}(2p + |\ell| + 1) \cdot \cos^{-1}(\sqrt{g_1 g_2})],$$

with $q >> 1$, $p = 0, 1, 2, \ldots$, and $\ell = 0, \pm 1, \pm 2, \ldots$. A mode with index ℓ has an associated spatial variation $e^{i\ell\theta}$. Mode stability requires that $0 \leq g_1 g_2 \leq 1$, the equality signs marking conditional stability.

(a - 2pts) Suppose a symmetric concentric optical resonator has mirrors of curvature 0.75 m. Give numbers for the length of the cavity and the free spectral range of the cavity in MHz.

(b - 2pts) Prove that a symmetric concentric resonator is stable if the cavity length is slightly less than the precise cavity length required for a concentric cavity, but is unstable if the length is slightly larger.

(c - 2pts) Suppose that a two-mirror optical cavity has a frequency difference between any two adjacent radial modes, all other mode indices being fixed, that is equal to one-half of the free spectral range. If the two mirrors have the same radius of curvature $R > 0$, what are the permissible values of the cavity length L relative to the mirror curvature?

(d - 2pts) For a confocal resonator consider the family of modes with indices (q, p, ℓ) for which $(2p + |\ell|) = M$ is equal to an even integer M. Show that this family of modes coincides in frequency with the mode with indices $(q + M/2, 0, 0)$. (Confocal resonators are special due to this degeneracy).

(e - 2pts) Draw the transverse intensity spot patterns for the $\text{LG}_{p\ell}$ transverse modes with $(p, \ell) = (0, 0), (2, 0), (0, 2)$.